

# Electrostatic interaction of a pointlike charge with a wormhole

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**Abstract.** A pointlike static (or quasistatically moving) electric charge  $q$  is considered in the spacetime which is a wormhole connecting two otherwise Minkowskian spaces. The electrostatic force acting on the charge is found to be a sum of two terms. One of them is uniquely determined by the value of  $q$  and the geometry of the wormhole. The other has the Coulomb form and is proportional to a freely specifiable parameter (the “charge of the wormhole”). These terms are interpreted, respectively, as the self-force and the force exerted on the charge by the wormhole. The self-force is found explicitly in the limit of vanishing throat length. The result differs from that obtained recently by Khusnutdinov and Bakhmatov.

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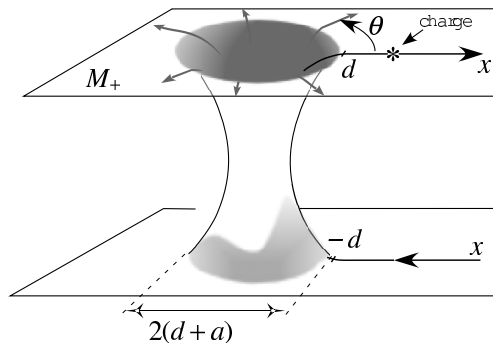
## 1. Introduction

What electric force (if any) acts on a pointlike charge at rest outside a wormhole, if there are no more charges in the space? This question is of interest by, at least, two reasons. The first is its relation to the famous concept of “charge without charge” [1]. Suppose, in a flat region of a spacetime we observe the electric field

$$\mathbf{E} = Q\mathbf{r}/r^3, \quad r > r_0. \quad (1)$$

From this we need not conclude that the field is generated by a charge (sitting, say, at  $r = 0$ ). It may well happen that there is a wormhole mouth inside the sphere  $r = r_0$  (so that the “coordinate”  $r$  does not, in fact, take the zero value) and the field force lines do not terminate at all, see figure 1. Which suggests that maybe there is no charge — as “substance” — in nature and the electromagnetic field is described by the source-free Maxwell equations, while all elementary “charges” are, in fact, mouths of wormholes<sup>‡</sup>. In developing such a theory it would be important to know how strong the resemblance is between a wormhole threaded by force lines and a pointlike charge  $Q$ . The flux conservation together with the spherical symmetry guarantees that (1) is valid in both cases. So, one might think that, as long as we restrict ourselves to the region

<sup>‡</sup> Throughout the paper we regard the matter supporting the wormhole as electrically neutral.



**Figure 1.** A section  $\varphi = \text{const}$  of  $M^{(3)}$ . The divergent gray lines are the force lines of a source-free field which “imitates” the Coulomb field for an observer outside the sphere  $r = d + a$ .

$r > r_0$ , the resemblance is perfect. As we shall see, however, *this is not the case*: the force  $\mathbf{F}$  experienced by a finite charge  $q$  put in a point  $p_*$  outside the wormhole would *not* be just the Coulomb force  $\mathbf{F}_C = qQ\mathbf{r}(p_*)/r^3(p_*)$  (moreover,  $\mathbf{F}_C$  may turn out to be a negligible part of  $\mathbf{F}$ ).

Another reason of interest in finding  $\mathbf{F}$  is the possible existence of macroscopic traversable wormholes. At the moment the only observational restriction on their abundance has been obtained on the basis of unusual lensing properties of negative mass [2] and is valid only for a very special type of wormholes. To improve the situation it would be desirable, of course, to know more about physical effects involving wormholes. The consideration of the electrostatic problem in the wormhole background can be viewed as a step in that direction. An interesting, in this sense, result of this paper is that self-interaction leads to appearance of the attraction infinitely growing (in the approximation of infinitely short throat) as the charge approaches a mouth of a wormhole. Tempted by the resemblance between the electrostatic and (Newton’s) gravitational forces — which differ in the *sign*, though — one might speculate therefore that wormholes are possible which are macroscopic and static, but nevertheless non-traversable for massive bodies.

Our analysis will be confined to a simplest wormhole:

$$ds^2 = -dt^2 + dx^2 + r^2(x)(d\theta^2 + \sin^2\theta d\varphi^2) \quad (2a)$$

$$x \in \mathbb{R}, \quad r \in C^\infty, \quad r(-x) = r(x), \quad r > 0, \quad r|_{x>d} = x + a. \quad (2b)$$

The wormhole is obviously static and spherically symmetric. Each its spacelike section  $t = \text{const}$  — we shall denote such sections by  $M^{(3)}$  — is a pair of flat three-dimensional spaces  $M_+$  and  $M_-$  (they are defined by the inequalities  $x > d$  and  $x < -d$ , respectively, and either is just the Euclidean space minus a ball of radius  $d + a$ ) connected with a ‘tunnel’, see figure 1.

It is well known [3, problem 14.16] that in curved spacetime the Maxwell equations written for the vector-potential  $A^i$  have, in the general case, two non-equivalent versions. Fortunately,  $R_i^0 = 0$  for our metric and the difference does not lead to any ambiguity in the equation on  $\Phi = A^0$ . It reads:

$$\Phi_{,a}{}^{;a}(p, p_*) = -4\pi q \delta(p - p_*), \quad p \in M^{(3)}, \quad (3a)$$

where  $a = 1, 2, 3$  and the derivatives are by the coordinates of  $p$  (not  $p_*$ ). Equation (3a) can be solved by standard methods, see the following section, but there are two problems in finding the force  $\mathbf{F}$  from  $\Phi$ :

A. The solutions of (3a) diverge in  $p_*$ , where the force is to be found, and thus one needs a “renormalization” procedure to derive a meaningful and finite value for the force. The problem is quite hard in the general case, see, e. g., [4, 5], and references therein, but in the case under discussion  $p_*$  is restricted to  $M_+$ , where the procedure is trivial due to flatness: to obtain the renormalized solution  $\Phi^{\text{ren}}$  one simply subtracts the Coulomb part from  $\Phi$ , see [4, 6].

B. The more serious problem is that (3a) has too many solutions: if some  $\Phi_1$  solves (3a) then so also does

$$\Phi_1(p, p_*) + f(p_*)Z(p),$$

where  $f$  is arbitrary and the “source-free” potential  $Z$  is an arbitrary harmonic function. In the ordinary electrostatics the problem is solved by requiring the electric field to fall at infinity

$$\Phi_{,i} \rightarrow 0 \quad \text{at} \quad r_* \equiv r(p_*) = \text{const}, \quad r(p) \rightarrow \infty, \quad (3b)$$

which physically means that we are not interested in field configurations with infinite energies. We adopt the restriction (3b) too, but in our case this does not fix the problem, because in  $M^{(3)}$  there *are* non-zero harmonic functions satisfying (3b). Thus, the (absolute value of) the force experienced by a pointlike charge near the wormhole is *arbitrary* and the question posed in the beginning of the paper has no meaningful answer. To overcome this problem I introduce “the charge” of the wormhole defined — up to the factor  $4\pi$  — as the flux of  $\mathbf{E}$  through the throat and prove (see the proposition in the following section) that *for the wormhole of a given charge  $Q$  the solution of (3) is unique* up to an additive constant. The solution depends on  $Q$  in quite a natural way:

$$\Phi^{\text{ren}}(q, p, p_*) = Q/r(p) + \Phi_{\text{sf}}(q, p, p_*) + \text{const}. \quad (*)$$

[cf. (17)]. The desired force acting on the charge is found now by, first, differentiating this expression by the coordinates of  $p$  and then setting  $p = p_*$ :

$$\mathbf{F}(r) = qQ(r)\mathbf{r}/r^3 - q\nabla\Phi_{\text{sf}}(r, r) \quad (4a)$$

(as before,  $\nabla$  in this expression acts on the first argument).

Formally, equation (4a) solves the problem in discussion [an explicit expression for  $\Phi_{\text{sf}}$  is given by (17)]. It is, however, of little practical use yet. Indeed, by  $\mathbf{F}(p)$  one normally understands the dependence of the force on position of the charge when everything except the position is assumed to be fixed. But this latter (perhaps, somewhat vague) condition in no way enters the derivation of (4a) and the function  $Q(r)$  is therefore *arbitrary*. To fix it suppose that the charge is transported (quasistatically, so that the radiation can be neglected) from  $p_*$  to some other point  $p_{**} \in M_+$ . In section 3, I argue that  $Q$  in such a case will remain unchanged  $Q(r_*) = Q(r_{**})$ , which conservation justifies the name “charge”. So, (4a) must be complemented with

$$Q(r) = \text{const}, \quad (4b)$$

which accomplishes the task.

The structure of (4) with  $Q$  independent of  $p_*$  and  $\Phi_{\text{sf}}$  independent of  $Q$  suggests interpretation of the first term in (4a) as the force exerted on the charge by the source-free field, or by the wormhole. And the second term is naturally interpreted as the self-force.

*Note.* Recently, Khusnutdinov and Bakhmatov [5] have found special solutions — let us denote them by  $\Phi_{\text{KB}}^{(1)}$  and  $\Phi_{\text{KB}}^{(2)}$  — of equations (3) for  $r = \sqrt{a^2 + x^2}$  and  $r = a + |x|$ , respectively (later  $\Phi_{\text{KB}}^{(1)}$  was refound by Linet [7], who used a different method). Neither of those  $r(x)$  satisfies (2b), but the main problem with finding the self-force [which is how to identify the self-interaction potential among the infinitely many solutions of (3)] is the same as in our case. Correspondingly, as explained above, the quantity  $-q\nabla\Phi_{\text{KB}}^{\text{ren}(i)}(r, r)$  need not be the self-force. And, indeed, calculating the flux of  $\nabla\Phi_{\text{KB}}^{\text{ren}(1)}$  through the sphere  $x = \text{const}$  one finds that it *depends on*  $p_*$ , see, e. g., [7, (20)] and the sentence below it. Likewise, for the wormhole of the second type the comparison of our formula (20) with that for  $G^{\text{ren}}$  in [5] gives the flux  $-qa/(2r_*)$ . So,  $\mathbf{F} \equiv -q\nabla\Phi_{\text{KB}}^{\text{ren}(i)}(p_*, p_*)$  is *not* the self-force, but rather another, much less meaningful, quantity — the force acting on the pointlike charge located in  $p_*$  in the presence of a wormhole with the charge  $Q(p_*)$ .

## 2. The multipole expansion

In this section we establish the uniqueness of the solution of equation (3) up to the term  $Q\rho/r + \Phi_0$ , where  $\rho$  is a certain function of  $r$  (specified below), while  $Q$  and  $\Phi_0$  do not depend on  $r$ .

We begin by rewriting equation (3a) in the coordinate form

$$\begin{aligned} \left[ \partial_x^2 + \frac{2r'}{r} \partial_x + \frac{1}{r^2} (\partial_\theta^2 + \cot \theta \partial_\theta + \sin^{-2} \theta \partial_\varphi^2) \right] \Phi \\ = -\frac{4\pi q}{r^2 \sin \theta} \delta(\varphi) \delta(\theta) \delta(x - x_*) \end{aligned}$$

(we have set  $\varphi_* = \theta_* = 0$ , which obviously does not lead to any loss of generality).

Expanding

$$\Phi = \sum_{l=0}^{\infty} \sum_{m=-l}^l \phi_l^{(m)}(x) Y_l^m(\varphi, \theta),$$

where  $Y_l^m$  are spherical functions [8]

$$Y_l^m(\varphi, \theta) \equiv \sqrt{\frac{2l+1}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}} \mathcal{P}_l^{|m|}(\cos \theta) e^{im\varphi},$$

$$\mathcal{P}_l^m(\mu) \equiv (1-\mu^2)^{\frac{m}{2}} \frac{d^m}{d\mu^m} \mathcal{P}_l(\mu)$$

( $\mathcal{P}_l$  are the Legendre polynomials) one gets

$$\begin{aligned} \sum_{l=0}^{\infty} \sum_{m=-l}^l \left[ \partial_x^2 + \frac{2r'}{r} \partial_x - \frac{l(l+1)}{r^2} \right] \phi_l^{(m)}(x) Y_l^m(\varphi, \theta) \\ = -\frac{4\pi q}{r^2 \sin \theta} \delta(\varphi) \delta(\theta) \delta(x - x_*) \end{aligned} \quad (5)$$

Multiply both sides of (5) by  $r Y_l^{m'*} \sin \theta$  and integrate over  $\varphi$  and  $\theta$ . The result ( $Y_l^{m'}$  are orthonormal on the sphere) is

$$\left[ \partial_x^2 - \left( \frac{r''}{r} + \frac{l(l+1)}{r^2} \right) \right] r \phi_l^{(m)}(x) = -\frac{4\pi q}{r} \delta(x - x_*) Y_l^{m*}(0, 0),$$

It is convenient to treat the cases of zero and non-zero  $m$  separately, because

$$Y_l^0(0, 0) = \sqrt{\frac{2l+1}{4\pi}}, \quad Y_l^m(0, 0) = 0, \quad m \neq 0$$

So, we define

$$v_l \equiv \sqrt{\frac{2l+1}{4\pi}} r \phi_l^{(0)}, \quad v_{l,m} \equiv r \phi_l^{(m)}, \quad m \neq 0.$$

For  $v_l$  we have

$$\left[ \partial_x^2 - \left( \frac{r''}{r} + \frac{l(l+1)}{r^2} \right) \right] v_l = -\frac{2l+1}{r_*} q \delta(x - x_*) \quad (6)$$

while  $v_{l,m}$  irrespective of  $m$  must solve the equation

$$\left[ \partial_x^2 - \left( \frac{r''}{r} + \frac{l(l+1)}{r^2} \right) \right] z(l, x) = 0. \quad (7)$$

Thus, the solution of (3) is the function

$$\Phi = \frac{1}{r} \sum_{l=0}^{\infty} v_l(x) \mathcal{P}_l(\cos \theta) + \frac{1}{r} \sum_{l=1}^{\infty} \sum_{|m|=1}^l v_{l,m}(x) Y_l^m(\varphi, \theta), \quad (8)$$

where  $v_l$  and  $v_{l,m}$  are the solutions, respectively, of (6) and (7) which [because of (3b)] grow at  $|x| \rightarrow \infty$  not faster than  $|x|$ .

To proceed note that in the flat regions  $M_{\pm}$  the term with  $r''$  vanishes in (7) and the equation is easily solved: the solution is a superposition of  $r^{-l}$  and  $r^{l+1}$ .

*Notation* By  $z_-$  and  $z_+$  we denote the solutions of (7) which are equal to  $r^{-l}$  at, respectively,  $x < -d$  and  $x > d$ . And  $z_e, z_o$  are the solutions of (7) defined by the initial data

$$z_e(0) = 1, \quad z'_e(0) = 0, \quad z_o(0) = 0, \quad z'_o(0) = 1$$

Evidently  $z_e$  and  $z_o$  are even and odd, respectively, and any solution of (7) is their linear combination.

*Proposition.* If  $z(l, x)$  is a solution of (7) with  $l > 0$ , the function  $r^{-1}z$  grows unboundedly as  $x \rightarrow (-)\infty$ .

*Proof.* We start with the observation that if a solution  $z$  of (7) satisfies the condition

$$z(x_0) > 0, \quad W[z, r](x_0) \geq 0, \tag{9a}$$

where  $W$  is the Wronskian  $W[f_1, f_2] = f'_1 f_2 - f_1 f'_2$ , then

$$z'/z > 0 \quad \text{and} \quad z/r \text{ grows} \quad \text{at } x > x_0. \tag{9b}$$

Indeed, rewrite (7) as

$$W'[z, r] = \frac{l(l+1)}{r} z. \tag{10}$$

Integrating this equation one gets

$$\frac{z'}{z} - \frac{r'}{r} = \frac{1}{rz} W[z, r](x_0) + \frac{l(l+1)}{rz} \int_{x_0}^x \frac{z \, dx}{r}. \tag{11}$$

Due to (9a) the r. h. s. is positive at least up to  $x_1$ , where  $x_1$  is  $\infty$ , if  $z(x)$  has no zeroes, and the first zero of  $z$  otherwise. Thus,  $z'/z > 0$  and  $z/r$  grows at  $x \in (x_0, x_1)$ . The latter means, in particular, that  $x_1$  cannot be finite (because if it were,  $r(x_1)$  would have been less than  $z(x_1) = 0$ ), which proves (9b).

Now note that both  $z = z_e$  and  $z = z_o$  satisfy (9a) with  $x_0$  equal to zero in the former case and to some (sufficiently small) positive number in the latter. So,  $z_{e(o)}/r$  grows at all  $x > x_0$  and hence,  $z_{e(o)}$  cannot be proportional to  $r^{-l}$  at large  $x$ . Consequently,

$$z_{e(o)}(x) \sim c_2 r^{l+1}, \quad x \rightarrow \infty, \quad c_2 \neq 0.$$

The same is true for  $x \rightarrow -\infty$ , since  $z_{e(o)}$  is even (odd). And, finally, it is true, when  $x \rightarrow \infty$  or  $x \rightarrow -\infty$ , for *every*  $z$  because any of them is a superposition of  $z = z_e$  and  $z = z_o$ .  $\square$

*Corollary 1.* If  $l > 0$ , the solutions  $z_+$  and  $z_-$  are linearly independent.

*Corollary 2.* The second term in the r. h. s. of (8) is zero.

*Corollary 3.* Denote by  $\vartheta$  the Heaviside step function. Then at  $l > 0$  the function

$$v_l = -\frac{(2l+1)q}{r_*} \frac{\vartheta(x-x_*)z_+(l,x)z_-(l,x_*) + \vartheta(x_*-x)z_-(l,x)z_+(l,x_*)}{W[z_+, z_-]} \quad (12a)$$

is the unique solution of (6) that grows slower than  $r^{l+1}$  at the infinities.

Now let us turn to the case  $l = 0$ . Equation (7) [as seen from (10)] transforms into  $W'[z, r] = 0$ . This gives

$$(z/r)' = C/r^2,$$

where  $C$  is an arbitrary constant. Thus,  $z(0, x)$  is a linear combination of  $r$  and  $\rho$

$$\rho(x) \equiv \frac{r(x)}{r(d)} - r(x) \int_d^x \frac{dx}{r^2(x)}.$$

In this case  $z_+$  is proportional to  $z_-$  and the formula (12a) does not define  $v_0$ . The latter, however, can be easily found by using — as independent solutions of the homogeneous equation (7) — the functions  $\rho$  and  $r$  instead of  $z_+$  and  $z_-$  (note that  $W[r, \rho] = 1$ ):

$$v_0 = \frac{q}{r_*} \left( \vartheta(x-x_*)\rho(x)r_* + \vartheta(x_*-x)r\rho_* \right) + Q\rho + \Phi_0 r \quad (12b)$$

Here  $\rho_* = \rho(x_*)$  and  $Q, \Phi_0$  are arbitrary, but do not depend on  $r$ .

Summing up,

$$\Phi = \frac{1}{r} \sum_{l=0}^{\infty} v_l(x) \mathcal{P}_l(\cos \theta), \quad (13)$$

where  $v_l(x)$  are given by formulae (12).

### 3. Self-interaction

Equation (13) gives, in principle, the electrostatic field of a pointlike charge in the wormhole background. However, as mentioned in the Introduction, to find the *force* acting on the charge it remains to cope with the fact that the field diverges in the point  $p_*$  where the charge is located. To this end we take advantage of the fact that  $p_*$  is in a flat part of the wormhole (let it be  $M_+$ , for definiteness). In this region we define the potential (for the second equality see, e. g., [8, (II 2.13)])

$$\begin{aligned} \Phi_{\text{Eucl}}(p, p_*) &\equiv \frac{q}{|p, p_*|} = \\ &= \frac{1}{r} \sum_{l=0}^{\infty} q [\vartheta(x-x_*)(r_*/r)^l + \vartheta(x_*-x)(r/r_*)^{l+1}] \mathcal{P}_l(\cos \theta), \end{aligned} \quad (14)$$

where  $|p, p_*|$  is the distance between  $p$  and  $p_*$  in the space  $\mathbb{E}^3$ , obtained by gluing a usual Euclidean ball of radius  $d$  to  $M_+$ . From the usual electrostatics we know that the field  $-\nabla \Phi_{\text{Eucl}}$  exerts no force on the charge. So, in finding the self-force we are only interested in the difference

$$\Phi^{\text{ren}} \equiv \Phi - \Phi_{\text{Eucl}} \quad (15)$$

(which is defined, of course, only in  $M_+$ ). It is  $\Phi^{\text{ren}}$  that plays the rôle of the “external field”, i. e. the force acting on the charge is  $\mathbf{F} = -q\nabla\Phi^{\text{ren}}(p_*, p_*)$ .

To rewrite the expression (13) for  $\Phi$  in a more convenient form let us substitute the equalities (in fact, the second one is a definition of  $\alpha_l$ )

$$z_+(x, l) = r^{-l}, \quad z_-(x, l) = C(r^{l+1} + \alpha_l r^{-l}), \quad \text{at } l > 0, x > d, \quad (16)$$

into (12a):

$$v_l = q[\vartheta(x - x_*)(r_*/r)^l + \vartheta(x_* - x)(r/r_*)^{l+1} + \frac{\alpha_l}{r_*}(rr_*)^{-l}],$$

$$\text{at } l > 0, \quad x, x_* > d$$

Substituting this together with an obvious (notice that  $\rho(x) = 1$  at  $x > d$ ) equality

$$v_0 = q[\vartheta(x - x_*) + (r/r_*)\vartheta(x_* - x)] + Q + r\Phi_0, \quad \text{at } x, x_* > d.$$

into (13) and, then, the result — combined with (14) — into (15), we finally obtain

$$\Phi^{\text{ren}}(p, p_*) = \Phi_{\text{sf}}(p, p_*) + \Phi_{\text{wh}}(p, p_*), \quad p, p_* \in M_+, \quad (17a)$$

where

$$\Phi_{\text{sf}} \equiv q \sum_{l=1}^{\infty} \alpha_l (rr_*)^{-l-1} \mathcal{P}_l(\cos \theta), \quad \Phi_{\text{wh}} \equiv Q/r + \Phi_0. \quad (17b)$$

*Note* In the region under consideration (i. e., at  $x, x_* > d$ )  $\Phi_{\text{sf}}$  is smooth.

*Proof.* By definition [see, (16)]

$$\begin{aligned} \text{at } x = d \quad \alpha_l &= -\frac{(z_- r^{-l-1})' r^{2(l+1)}}{(2l+1)C} = \\ &= -\frac{(z_- r^{-l-1})' r^{2(l+1)}}{(2l+1)} \frac{(2l+1)r^{2l}}{(z_- r^l)'} = \frac{l/r - z_-'/z_- + 1/r}{l/r + z_-'/z_-} r^{2l+1}(d) \end{aligned} \quad (18)$$

On the other hand,  $z_-$  satisfies the condition (9a) with  $x_0 = -d$ . Hence, by (9b),  $z_-'/z_-$  is positive at  $x = d$ . It follows then from (18) that at  $l \rightarrow \infty$

$$\alpha_l = A_l r^{2l+1}(d), \quad A_l = O(1)$$

and

$$\Phi_{\text{sf}}(x, x_*) = q \sum_{l=1}^{\infty} \frac{A_l}{d+a} \left[ \frac{r(d)}{r(x)} \frac{r(d)}{r(x_*)} \right]^{l+1} \mathcal{P}_l(\cos \theta).$$

Obviously for any  $x_1 > d$  the series converges uniformly on  $[x_1, \infty)$  and so do all the series obtained from this one by termwise differentiation in  $r$ .  $\square$

We interpret  $\Phi_{\text{sf}}$  and  $\Phi_{\text{wh}}$  as the parts of  $\Phi^{\text{ren}}$  generated by the charge and by the wormhole, respectively. To justify this interpretation note that 1)  $\Phi_{\text{sf}}$ , for a given  $p$ , depends only on  $q$  and  $p_*$  and 2)  $\Phi_{\text{wh}}$ , in contrast, does not depend on  $p_*$  in the following sense. Suppose the charge is at rest up to some moment  $t_0$  and is then quasistatically moved from  $p_*(t_0)$  to some  $p_*(t_1) \in M_+$ , where — at the moment  $t_1$  — is again put to



rest. Let  $t_2$  be a moment when at small  $|x|$  the disturbance in the potential caused by the motion of the charge has already settled down and the potential became constant (in time)<sup>§</sup>. Then at times  $t > t_2$  in the vicinity of the wormhole the equations (17) remain valid with  $r_*(t_0)$  replaced by  $r_*(t_1)$  and with the *same*  $Q$ .

*Proof.* Indeed, at  $t > t_2$  the flux of  $\nabla\Phi$  through the sphere  $x = d$  is  $\mathcal{F}(d, t_2) = -4\pi Q(t_2)$ , because neither  $\Phi_{\text{Eucl}}$ , nor  $\Phi_{\text{sf}}$  give any contribution to it. At the same time, there is a sphere  $x = D > x_*$  such that  $\mathcal{F}(D, t_2) = -4\pi[Q(t_0) + q]$ , because if  $D > x_* + c(t - t_0)$ , the field is not disturbed there yet. Thus the flux  $\mathcal{F}_B(t_2)$  through the boundary of the layer  $\{d < x < D, t = t_2\}$  is  $4\pi[Q(t_2) - Q(t_0) - q]$ . On the other hand, the total charge inside the layer has not changed and hence  $\mathcal{F}_B(t_2) = \mathcal{F}_B(t_0) = -4\pi q$  by the Gauss theorem. So,  $Q(t_2) = Q(t_0)$ .  $\square$

#### 4. Short wormhole

It is seen from formulae (17) that the force acting on a charge depends on the form of the wormhole — the information about the form being encoded in the coefficients  $\alpha_l$ . But today we have no reason to consider any particular form as more realistic than any other. So, it would be interesting to find a form-independent effect. To this end we consider in this section the limit  $d \rightarrow 0$  for the wormhole (2) with  $a > 0$ . In doing so we allow the the throat to be *arbitrary*, the only additional requirement on  $r(x)$  being

$$r' < c_r \quad \forall d \quad (19)$$

( $c_r$  is a constant), which, among other things, guarantees that  $r(0) \rightarrow a$ .

Let us, first, present  $z_-$  as the solution of the following differential equation (which does not contain the large quantity  $r''$ )

$$\begin{aligned} \left[ \partial_x^2 + 2r'r^{-1}\partial_x - l(l+1)r^{-2} \right] y &= 0, \\ y(-d) &= (d+a)^{-l-1}. \end{aligned}$$

Here the first line is simply (7) in terms of  $y \equiv z_-/r$ , while the second follows from the definition of  $z_-$ . The coefficients of the equation by (19) are uniformly (by  $d$ ) bounded, so at a *fixed*  $l$  and  $d \rightarrow 0$

$$\ln' y(-d) \rightarrow \ln' y(d).$$

Hence,  $\ln' z(-d) - \ln' r(-d) \rightarrow \ln' z(d) - \ln' r(d)$  and thus

$$\ln' z_-(d) \rightarrow \ln' z_-(-d) + 2/a \rightarrow (l+2)/a$$

(recall that  $z_- = (a-x)^{-l}$  at  $x = -d$ ). On the other hand, by (16)

$$\ln' z_-(d) = \frac{(l+1)r^l - l\alpha_l r^{-l-1}}{r^{l+1} + \alpha_l r^{-l}} \Big|_{r=a} = \frac{(l+1)a^l - l\alpha_l a^{-l-1}}{a^{l+1} + \alpha_l a^{-l}},$$

<sup>§</sup> The existence of such a moment is an *assumption*, even though a very plausible one. If a wormhole is such that in its vicinity the electro-magnetic waves caused by stirring the charge do not dissipate with time, one probably cannot develop electrostatics in that spacetime at all.

combining which with the equation above we find in the limit  $d \rightarrow 0$

$$\alpha_l = -\frac{a^{2l+1}}{2(l+1)}.$$

Thus, asymptotically,

$$\begin{aligned}\Phi^{\text{ren}} &\sim -q \sum_{l=1}^{\infty} \frac{a^{2l+1}}{2(l+1)} (rr_*)^{-l-1} \mathcal{P}_l(\cos \theta) + \frac{Q}{r} + \Phi_0 \\ &= -\frac{q}{2a} \sum_{l=2}^{\infty} \frac{1}{l} (a^2/rr_*)^l \mathcal{P}_{l-1}(\cos \theta) + \frac{Q}{r} + \Phi_0.\end{aligned}$$

In particular, at  $\theta = 0$

$$\begin{aligned}\Phi^{\text{ren}} &\sim -\frac{q}{2a} \sum_{l=2}^{\infty} \frac{1}{l} (a^2/rr_*)^l + \frac{Q}{r} + \Phi_0 \\ &= \frac{q}{2a} \left[ \ln\left(1 - \frac{a^2}{rr_*}\right) + \frac{a^2}{r_*r} \right] + \frac{Q}{r} + \Phi_0\end{aligned}\quad (20)$$

and the electric field on the axis is

$$-\Phi_{,x} = -(\Phi^{\text{ren}} + \Phi_{\text{Eucl}})_{,x} \sim q \frac{r - r_*}{|r - r_*|^3} + \frac{qa^3}{2r_*r^2(a^2 - rr_*)} + \frac{Q}{r^2}.$$

Thus, asymptotically, in the presence of an infinitely short wormhole with the radius  $a$  a pointlike charge  $q$  experiences the (radial) force

$$F(r_*) = -q\Phi_{,x}^{\text{ren}} = -\frac{q^2a^3}{2r_*^3(r_*^2 - a^2)} + \frac{qQ}{r_*^2}.$$

Its first term — the self-force  $\mathbf{F}_s$  — can be presented, if desired, in the form

$$\mathbf{F}_s(r_*) = -\nabla U(r_*), \quad U(r_*) = \frac{q^2}{4a} \left( \ln[1 - (a/r_*)^2] + (a/r_*)^2 \right)$$

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